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A Variation of Rational L_1 Approximation*

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A new approximating method proposed by A. Pinkus and O. Shisha is extended to rational approximation. The existence, characterization, uniqueness, strong uniqueness, and continuity of best approximation are established. © 1990 Academic Press, Inc.

NOTATION

For $f \in C[0, 1]$, the measure $|||\cdot|||$ introduced by Pinkus and Shisha [2] is

$$|||f||| = \sup_{0 \leqslant a \leqslant b \leqslant 1} \left\{ \left| \int_a^b f\,dx \right| : f(x) > 0 \text{ on } (a, b) \text{ or } f(x) < 0 \text{ on } (a, b) \right\}.$$

With this measure, Pinkus and Shisha have studied best approximation from the set of algebraic polynomials of degree $\leqslant n$, and have established some remarkable results. Set

$$R_m^n := \{ p/q : p \in P_n,\ q \in P_m,\ p/q \text{ is irreducible},$$
$$q > 0 \text{ on } [0, 1] \},$$

where P_n denotes the set of all real algebraic polynomials of degree $\leqslant n$. For $f \in C[0, 1]$, one can consider the following problem: find $r_0 \in R_m^n$ such

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that $|||f-r_0||| = \inf\{|||f-r||| : r \in R_m^n\}$. Any such r_0 is called a best approximation to f from R_m^n (with respect to $|||\cdot|||$).

In this paper we consider the basic questions of existence, characterization, uniqueness, and continuity of best approximation, and get some interesting results analogous to the well-known theorems for the Chebyshev norm $\|\cdot\|$ (throughout this paper $\|\cdot\|$ denotes $\|\cdot\|_\infty$).

1. Existence

Using the method in the proof of Theorem 2.5 of [2], one can obtain:

LEMMA 1.1. *Assume that for* $k = 1, 2, ..., f_k \in C[0, 1]$, $p_k \in P_n$, *and* $\{\|f_k\|\}$ *is bounded. Then the sequence* $\{\|p_k\|\}$ *is bounded whenever* $\{|||f_k - p_k|||\}$ *is bounded.*

We also need:

LEMMA 1.2. *Assume that for* $k = 1, 2, ..., r_k \in R_m^n$, $f_k \in C[0, 1]$, *and* $\{\|f_k\|\}$ *is bounded. If* $|||r_k||| \to +\infty$, *then*

$$\underline{\lim}_{k \to \infty} |||f_k - r_k|||/|||r_k||| \geqslant \frac{1}{s+1},$$

where $s = \max\{m, n\}$.

Proof. Assume that for $k = 1, 2, ...$, an open interval $I_k = (a_k, b_k)$ in $[0, 1]$ is such that for some $e_k = 1$ or -1, fixed, $e_k r_k > 0$ on I_k and $e_k \int_{I_k} r_k \, dx = |||r_k|||$. Let $\|f_k\| \leqslant M$ for $k = 1, 2, ...$ Without loss of generality assume that $|||r_k||| > (s+2)M$ for all $k > 0$. Then $e_k M - r_k$ has at most s zeros in $[0, 1]$, for otherwise $r_k = e_k M$ on $[0, 1]$. Hence

$$|||e_k M - r_k|||_{[a_k, b_k]} \geqslant \frac{1}{s+1} \int_{I_k} |e_k M - r_k| \, dx.$$

Assume that an open interval $\tilde{I}_k \subset I_k$ is chosen so that for some $\bar{e}_k = 1$ or -1, fixed,

$$\bar{e}_k (e_k M - r_k) > 0 \qquad \text{on } \tilde{I}_k \tag{1.1}$$

and

$$\bar{e}_k \int_{\tilde{I}_k} (e_k M - r_k) \, dx = |||e_k M - r_k|||_{[a_k, b_k]}. \tag{1.2}$$

One must have $\bar{e}_k = -e_k$, for otherwise,

$$|||r_k||| = e_k \int_{I_k} r_k \, dx \leqslant \int_{I_k} |e_k M - r_k| \, dx + M$$

$$\leqslant (s+1)\bar{e}_k \int_{\tilde{I}_k} (e_k M - r_k) \, dx + M < (s+2)M.$$

By virtue of (1.1) and (1.2) with $\bar{e}_k = -e_k$, we have $-e_k(f_k - r_k) \geqslant -M + e_k r_k = -e_k(e_k M - r_k) > 0$ on $\tilde{I}_k$ and

$$|||f_k - r_k||| \geqslant -e_k \int_{\tilde{I}_k} (f_k - r_k) \, dx$$

$$\geqslant -e_k \int_{\tilde{I}_k} (e_k M - r_k) \, dx = |||e_k M - r_k|||_{[a_k, b_k]}$$

$$\geqslant \int_{I_k} |e_k M - r_k| \, dx/(s+1) \geqslant (|||r_k||| - M)/(s+1).$$

Therefore $\underline{\lim}_{k \to \infty} |||f_k - r_k|||/|||r_k||| \geqslant 1/(s+1)$. The proof is completed.

Now we are ready to answer the question of existence.

THEOREM 1.3. *If $m \leqslant 1$, then for every $f \in C[0, 1]$ there is at least one best approximation to f from R^n_m.*

Proof. Let $E = \inf\{|||f - r||| : r \in R^n_m\}$. There is a sequence $\{r_k\}$ in R^n_m such that $|||f - r_k||| \to E$ as $k \to +\infty$.

Set $r_k = p_k/q_k$ for $k = 1, 2, \ldots$. Without loss of generality assume $\|q_k\| = 1$. So $|||q_k f - p_k||| \leqslant E + 1$ for sufficiently large k, and $\{\|p_k\|\}$ is bounded by Lemma 1.1. We can take a convergent subsequence of p_k and one of q_k (again denoted by p_k, q_k), say $p_k \to p$ and $q_k \to q$ as $k \to \infty$. Since $q \geqslant 0$ on $[0, 1]$, $q \in P_m$ and $m \leqslant 1$, q has at most one zero which is 0 or 1. It therefore follows that for every $\varepsilon > 0$, $p_k/q_k \to p/q$ uniformly on $[\varepsilon, 1 - \varepsilon]$.

Next we show that $p(0) = 0$ when $q(0) = 0$. Suppose to the contrary that $p(0) \neq 0$. Then there is a real c, $0 < c < 1$, and an integer $K > 0$ such that for some $e = 1$ or -1, fixed, $ep_k(x) > 0$ on $[0, c]$ for every $k > K$. Thus $ep_k(x)/q_k(x) > 0$ on $[0, c]$ and $|||r_k||| \geqslant e \int_{[0,c]} p_k/q_k \, dx$ for $k > K$. Hence $\{|||r_k|||\}$ and $\{|||f - r_k|||\}$ are all not bounded by Lemma 1.2. This is a contradiction. In the same way we have $p(1) = 0$ when $q(1) = 0$. Therefore whether or not $q(x)$ has a zero in $[0, 1]$, $r_0 = p/q$ is well defined in R^n_m.

It remains to show that r_0 is a best approximation to f. Assume that $(a, b) \subset [0, 1]$ is such that for some $e = 1$ or -1, fixed, $e(f - r_0) > 0$ on

(a, b) and $e\int_a^b (f-r_0)\,dx = \|f-r_0\|$. Thus for every ε with $0<\varepsilon<(b-a)/2$, one has that $e(f-r_k)>0$ on $[a+\varepsilon, b-\varepsilon]$ and

$$e\int_{a+\varepsilon}^{b-\varepsilon} (f-r_k)\,dx \leqslant \|f-r_k\|$$

for sufficiently large k. Letting $k\to\infty$ and $\varepsilon\to 0$, one has that $\|f-r_0\|\leqslant E$ and r_0 is a best approximation to f. The proof is completed.

For the remaining case, one has:

THEOREM 1.4. *If $m\geqslant 2$, then there exists a function f in $C[0,1]$ such that f does not have a best approximation from R_m^n.*

Proof. Define a function $f(x)$ in $C[0,1]$ such that

$$f(x)=\begin{cases}(n+2)/2 & \text{for}\quad x=1/(2n+4)\\ (-1)^k\,(n+2)/4 & \text{for}\quad x=(2k+1)/(2n+4),\ k=1,\ldots,n+1\\ 0 & \text{for}\quad x=i/(n+2),\ i=0,1,\ldots,n+2,\end{cases}$$

and $f(x)$ is linear in each of the remaining intervals. Set, for $k=0,1,\ldots,n+1$, $I_k=(k/(n+2), (k+1)/(n+2))$. Obviously $(-1)^k f>0$ on I_k for $k=0,1,\ldots,n+1$, and

$$(-1)^k\int_{I_k} f(x)\,dx=\begin{cases}\frac{1}{4} & \text{for}\quad k=0,\\ \frac{1}{8} & \text{for}\quad k=1,2,\ldots,n+1.\end{cases}$$

We claim that $\|f-r\|>\frac{1}{8}$ for every $r\in R_m^n$. In fact if $r=0$, then $\|f-r\|=\frac{1}{4}$. If $r\in R_m^n$ and $r\neq 0$, then there must be an interval I_s with $0\leqslant s\leqslant n+1$ such that $(-1)^s r\leqslant 0$ and $r\neq 0$ on I_s. Therefore $(-1)^s(f-r)\geqslant(-1)^s f>0$ on I_s and $\|f-r\|\geqslant(-1)^s\int_{I_s}(f-r)\,dx\geqslant\frac{1}{8}$.

Next we show that $\inf\{\|f-r\| : r\in R_m^n\}=\frac{1}{8}$. Set

$$r_k(x)=\frac{k}{4k^5(x-t)^2+1},$$

where $t=1/(2n+4)$. Then for $k=1,2,\ldots$, $r_k\in R_m^n$, $r_k>0$ on $[0,1]$, and

$$\lim_{k\to\infty} r_k(x)=\begin{cases}+\infty & \text{for}\quad x=t\\ 0 & \text{for}\quad x\neq t.\end{cases}\tag{1.3}$$

Hence in $(0, 1/(n+2))$ $f-r_k$ has four sign changes at the points $z_1<z_2<z_3<z_4$ with $z_1\to 0$ and $z_4\to 1/(n+2)$ as $k\to\infty$. Noting that

$(f-r_k)(t-1/k^2)=(f-r_k)(t+1/k^2)\to(2n+3)/4>0$ as $k\to\infty$, we also have $z_2\in(t-1/k^2, t)$ and $z_3\in(t, t+1/k^2)$ for sufficiently large k. Therefore

$$\int_0^{z_1} |f-r_k|\,dx\to 0 \qquad \text{as} \quad k\to\infty,$$

$$\int_{z_1}^{z_2}(f-r_k)\,dx=\int_{z_3}^{z_4}(f-r_k)dx<\int_0^t f\,dx=\tfrac{1}{8},$$

$$\int_{z_2}^{z_3}(r_k-f)\,dx<\int_{t-1/k^2}^{t+1/k^2} r_k\,dx\leqslant 2/k$$

and $|||f-r_k|||_{[0,z_4]}<\frac{1}{8}$ for sufficiently large k. Since $\int_0^1 |r_k|\,dx=o(1/k)$, it follows from (1.3) that $|||f-r_k|||_{[z_4,1]}=\frac{1}{8}+o(1/k)$. Hence $|||f-r_k|||=\frac{1}{8}+o(1/k)$ and

$$\inf\{|||f-r||| : r\in R_m^n\}=\tfrac{1}{8}.$$

The proof is completed.

2. Alternation Theorem

This section is devoted to the characterization of best approximants. We need some basic definitions.

Definition 2.1. For $f\in C[0,1]$, an extremal interval of f in $[0,1]$ is an open interval $I\subset[0,1]$, which for some $e=1$ or -1 (the signum of I) satisfies:

(1) $ef\geqslant 0$ on I,

(2) $e\int_I f(x)\,dx\geqslant |||f|||$.

Definition 2.2. For $f\in C[0,1]$, a maximal-definite interval of f in $[0,1]$ is an extremal interval $I=(\alpha,\beta)$ of f, which for $e=\operatorname{sign}(I)$ satisfies:

(i) if J is an open subinterval of $(0,1)$, $I\subset J$ and $ef\geqslant 0$ on J, then $f=0$ on $J\setminus I$;

(ii) there is no open, nonempty subinterval of I having α or β as an endpoint throughout which $f=0$.

As shown in [2], every f in $C[0,1]$ has finite maximal-definite intervals, and they are all mutually disjoint.

Now we are ready to establish:

Theorem 2.3. *For $f\in C[0,1]$, the irreducible rational function $r_0=p_0/q_0$*

is a best approximation to f from R_m^n if and only if $f-r_0$ has at least s alternating extremal intervals in $[0, 1]$; *i.e., $f-r_0$ has at least s extremal intervals $I_1 < I_2 < \cdots < I_s$ with*

$$\text{sign}(I_i) = -\text{sign}(I_{i+1}) \qquad \textit{for} \quad i = 1, 2, ..., s-1,$$

where $s = \max\{\partial p_0 + m, \partial q_0 + n\} + 2$ *and ∂p_0 denotes the degree of p_0.*

Proof. Assume that $I_1 < I_2 < \cdots < I_s$ are s alternating intervals of $f - r_0$ and $\text{sign}(I_1) = -e$. If there is $r_1 = p_1/q_1$ in R_m^n such that

$$|||f - r_1||| < |||f - r_0|||, \tag{2.1}$$

then for $i = 1, 2, ..., s$, there exists $x_i \in I_i$ satisfying

$$(-1)^i e(r_0 - r_1)(x_i) \leqslant 0. \tag{2.2}$$

Otherwise if for some i with $1 \leqslant i \leqslant s$, $(-1)^i e(r_0 - r_1) > 0$ on I_i, then $(-1)^i e(f - r_1) > (-1)^i e(f - r_0) \geqslant 0$ on I_i and

$$\begin{aligned} |||f - r_1||| &\geqslant (-1)^i e \int_{I_i} (f - r_1)\, dx \\ &> (-1)^i e \int_{I_i} (f - r_0)\, dx \geqslant |||f - r_0|||, \end{aligned}$$

a contradiction. From (2.2) and the fact that $\{p + q r_0 : p \in P_n, q \in P_m\}$ is a $(s-1)$-dimensional Chebyshev subspace (Lemma, [1, p. 162]), it follows that $q_1 r_0 - p_1 = 0$, i.e., $r_0 = r_1$. This contradiction completes the sufficiency of the theorem.

Assume that r_0 is a best approximation to f from R_m^n and all its maximal-definite intervals are

$$\begin{aligned} &I_1, I_2, ..., I_{m_1}, \\ &I_{m_1+1}, ..., I_{m_2}, \\ &\cdots \\ &I_{m_{t-1}+1}, ..., I_{m_t}, \end{aligned}$$

where $I_k < I_{k+1}$ for $1 \leqslant k \leqslant m_t - 1$, and for $e = 1$ or -1, fixed,

$$\text{sign}(I_i) = (-1)^j e \qquad \text{for} \quad m_j + 1 \leqslant i \leqslant m_{j+1}$$

with $0 \leqslant j \leqslant t-1$ and $m_0 = 0$. We show that $t \geqslant s$. If this is not the case, then for $j = 1, 2, ..., t-1$, a real x_j can be chosen so that $I_{m_j} < x_j < I_{m_j+1}$ and $(f - r_0)(x_j) = 0$. By virtue of Lemma of [1, p. 162] there are $p \in P_n$ and

$q \in P_m$ such that for $j = 0, 1, ..., t-1$, $(-1)^j e(p + qr_0) > 0$ on (x_j, x_{j+1}) with $x_0 = 0$ and $x_t = 1$. Since $(f - (p_0 + \lambda p)/(q_0 - \lambda q))(x_j) = 0$ for every $\lambda > 0$, it follows that

$$\begin{aligned} &|||f - (p_0 + \lambda p)/(q_0 - \lambda q)||| \\ &\quad = \max\{ |||f - (p_0 + \lambda p)/(q_0 - \lambda q)|||_{[x_j, x_{j+1}]} : 0 \leqslant j \leqslant t-1 \}. \end{aligned} \tag{2.3}$$

Noting that $q_0 - \lambda q > 0$ on $[0, 1]$ for sufficiently small $\lambda > 0$, we need only show that for $j = 0, 1, ..., t-1$,

$$|||f - (p_0 + \lambda p)/(q_0 - \lambda q)|||_{[x_j, x_{j+1}]} < |||f - r_0|||, \tag{2.4}$$

when $\lambda > 0$ becomes sufficiently small.

Suppose to the contrary that for some j with $0 \leqslant j \leqslant t-1$, (2.4) is not true. For $k = 1, 2, ...,$ there is $\lambda_k > 0$ such that $q_0 - \lambda_k q > 0$, $\lambda_k \to 0$, and $|||f - (p_0 + \lambda_k p)/(q_0 - \lambda_k q)|||_{[x_j, x_{j+1}]} \geqslant |||f - r_0|||$. Then for $k = 1, 2, ...,$ an interval $(a_k, b_k) \subset [x_j, x_{j+1}]$ can be chosen so that for some $e_k = 1$ or -1,

$$\begin{cases} e_k(f - (p_0 + \lambda_k p)/(q_0 - \lambda_k q)) > 0 & \text{on } (a_k, b_k) \\ e_k \displaystyle\int_{a_k}^{b_k} (f - (p_0 + \lambda_k p)/(q_0 - \lambda_k q))\, dx \geqslant |||f - r_0|||. \end{cases} \tag{2.5}$$

By passing to subsequences, if necessary, we may assume that $a_k \to a$, $b_k \to b$ as $k \to \infty$, and $e_k = \bar{e}$ for all k. Obviously $(a, b) \subset [x_j, x_{j+1}]$. Letting $k \to \infty$ in (2.5), one obtains

$$\bar{e}(f - r_0) \geqslant 0 \qquad \text{on } (a, b)$$

$$\bar{e} \int_a^b (f - r_0)\, dx \geqslant |||f - r_0|||.$$

Hence (a, b) must intersect some maximal-definite interval with the signum $\bar{e}$, and (2.3) implies that $\bar{e} = (-1)^j e$. It follows by (2.5) that $(-1)^j e(f - r_0) \geqslant (-1)^j e(f - (p_0 + \lambda_k p)/(q_0 - \lambda_k q)) + (-1)^j e\lambda_k(p + qr_0)/(q_0 - \lambda_k q) > 0$ on (a_k, b_k) and

$$\begin{aligned} |||f - r_0||| &\geqslant (-1)^j e \int_{a_k}^{b_k} (f - r_0)\, dx \\ &> (-1)^j e \int_{a_k}^{b_k} (f - (p_0 + \lambda_k p)/(q_0 - \lambda_k q))\, dx \\ &\geqslant |||f - r_0|||. \end{aligned}$$

This contradiction completes the proof of the theorem.

3. Uniqueness

Using the same method as that in the proof of the sufficiency of Theorem 2.3, one can obtain:

THEOREM 3.1. *Each f in $C[0, 1]$ has at most one best approximation from R^n_m.*

Furthermore a strong uniqueness theorem is presented.

THEOREM 3.2. *Assume that the ireducible rational function $r_0 = p_0/q_0$ is the best approximation to f from R^n_m and $(\partial p_0 - n)(\partial q_0 - m) = 0$. Then there exists a real $c > 0$ such that for every $r \in R^n_m$,*

$$|||f - r||| \geqslant |||f - r_0||| + c\, |||r - r_0|||. \tag{3.1}$$

Proof. If $r = r_0$, (3.1) is trivial. Set, for $r \in R^n_m$ with $r \neq r_0$, $\alpha(r) = (|||f - r||| - |||f - r_0|||)/|||r - r_0|||$. It is sufficient to show that $\alpha(r)$ has a positive infimum for all $r \in R^n_m$ with $r \neq r_0$. Suppose not. Then for $k = 1, 2, \ldots$, there exists $r_k = p_k/q_k$ in R^n_m such that $\|p_k\| + \|q_k\| = 1$ and $\alpha(r_k) \to 0$ as $k \to \infty$.

Since $r_k - r_0 \in R^{m+n}_{2m}$, by Lemma 1.2 we have that $\{|||r_k - r_0|||\}$ is bounded. Thus $|||f - r_k||| \to |||f - r_0|||$. Without loss of generality we may assume that $p_k \to p$ and $q_k \to q$ uniformly. By virtue of Theorem 2.3 there exist $m + n + 2$ open intervals $I_0 < I_1 \cdots < I_{m+n+1}$ in $[0, 1]$ and $e = 1$ or -1, fixed, such that for every i with $0 \leqslant i \leqslant n + m + 1$, $(-1)^i e(f - r_0) \geqslant 0$ on I_i and $(-1)^i e \int_{I_i} (f - r_0)\, dx \geqslant |||f - r_0|||$. We claim that for every k one can chose an integer $j(k)$ with $0 \leqslant j(k) \leqslant m + n + 1$ such that

$$e(-1)^{j(k)}(r_k - r_0) < 0 \qquad \text{on } I_{j(k)}. \tag{3.2}$$

If for some k this is not the case, then for each $j = 0, 1, \ldots, m + n + 1$, there exists a real $x_j \in I_j$ such that $e(-1)^j (r_k - r_0)(x_j) \geqslant 0$. Hence by Lemma of [1, p. 162] and Assertion of [4, p. 61] we have $r_k = r_0$, which contradicts the choice of r_k. Without loss of generality, assume $j(k) = \bar{m}$ for all k. Therefore, by virtue of (3.2), one has

$$\begin{aligned}\alpha(r_k)\, |||r_k - r_0||| &= |||f - r_k||| - |||f - r_0||| \\ &\geqslant (-1)^{\bar{m}} e \int_{I_{\bar{m}}} (f - r_k)\, dx - (-1)^{\bar{m}} e \int_{I_{\bar{m}}} (f - r_0)\, dx \\ &= e(-1)^{\bar{m}} \int_{I_{\bar{m}}} (r_0 - r_k)\, dx = \int_{I_{\bar{m}}} |r_k - r_0|\, dx.\end{aligned} \tag{3.3}$$

Since q has at most m zeros, a closed interval $\tilde{I} \subset I_{\bar{m}}$ can be chosen so that

$q>0$ on $\tilde{I}$. Hence by (3.3), $\int_{\tilde{I}} |p/q - r_0| \, dx = \lim_{k\to\infty} \int_{\tilde{I}} |r_k - r_0| \, dx = 0$ and $p/q = r_0$. By $(\partial p_0 - n)(\partial q_0 - m) = 0$ and Lemma 2 of [1, p. 165] we have $p = p_0$ and $q = q_0$ (assume $\|p_0\| + \|q_0\| = 1$). Thus $q>0$ and $q_k \geqslant \beta_1 > 0$ on [0, 1] for sufficiently large k. Let $\beta_2 = \inf\{\int_{I_m} |\tilde{p} + \tilde{q} r_0| \, dx : \tilde{p} \in P_n, \ \tilde{q} \in P_m, \ \|\tilde{p} + \tilde{q} r_0\| = 1\}$. Then $\beta_2 > 0$ and for sufficiently large k

$$\begin{aligned}
\alpha(r_k) \, |||r_k - r_0||| &\leqslant \int_{I_m} |r_k - r_0| \, dx \\
&= \int_{I_{\tilde{m}}} |p_k - q_k r_0|/|q_k| \, dx \geqslant \int_{I_{\tilde{m}}} |p_k - q_k r_0| \, dx \\
&\geqslant \beta_2 \, \|p_k - q_k r_0\| \geqslant \beta_1 \beta_2 \, \|r_k - r_0\| \\
&\geqslant \beta_1 \beta_2 \, |||r_k - r_0|||.
\end{aligned}$$

Since $|||r_k - r_0||| \neq 0$ the above equality contradicts the assumption that $\alpha(r_k) \to 0$. This contradiction completes the proof of the theorem.

4. Continuity

For $f \in C[0, 1]$, let $Tf \in R_m^n$ be the best approximation to f provided that one exists. The continuity of the operator T can be stated as follows:

THEOREM 4.1. *Assume that the irreducible rational function $r_0 = p_0/q_0$ is the best approximation to f_0 from R_m^n and $(\partial p_0 - n)(\partial q_0 - m) = 0$. Then for every $\varepsilon > 0$, there is a real $\delta > 0$ such that every f in $C[0, 1]$ with $\|f - f_0\| < \delta$ has a best approximation from R_m^n and $\|Tf - Tf_0\| < \varepsilon$.*

Proof. First we show that for every $\varepsilon > 0$, there exists a real $\delta_1 > 0$ such that $\|Tf - Tf_0\| < \varepsilon$ whenever $\|f - f_0\| < \delta_1$ and f has a best approximation Tf. Suppose to the contrary that for some $\varepsilon > 0$ there exists a sequence $\{f_k\}$in $C[0, 1]$ such that $\|f_k - f\| \to 0$ as $k \to \infty$, Tf_k exists for all k, and $\|Tf_k - Tf_0\| \geqslant \varepsilon$. Let $Tf_k = p_k/q_k$. Without loss of generality we assume that $\|p_0\| + \|q_0\| = \|p_k\| + \|q_k\| = 1$. By passing to subsequence, if necessary, assume that $p_k \to p$, $q_k \to q$, $|||f_k - Tf_k||| \to c$ as $k \to \infty$, and $\partial p_k = \partial p$, $\partial q_k = \partial q$ for every k. Since $q \geqslant 0$ on [0, 1], q can be decomposed as $q(x) = (x - z_1)^{s_1} \cdots (x - z_v)^{s_v} \tilde{q}(x)$, where $z_j \in [0, 1]$ for $j = 1, ..., v$, and $\tilde{q}(x) \neq 0$ on [0, 1]. For concreteness, assume $\tilde{q} > 0$ on [0, 1]. Using the method in the proof of Theorem 1.3, one can show that p must have the form $p(x) = (x - z_1)^{s_1} \cdots (x - z_v)^{s_v} \tilde{p}(x)$.

We consider the following two cases:

(i) $c \geqslant |||f - r_0|||$. By Theorem 2.3 for $k = 1, 2, ...,$ there are

$s = \max\{\partial p + m, \partial q + n\} + 2$ open intervals $I_1^{(k)} < \cdots < I_s^{(k)}$ and $e_k = 1$ or -1, fixed, such that for $i = 1, 2, ..., s$,

$$(-1)^i e_k (f_k - Tf_k) \geqslant 0 \qquad \text{on } I_i^{(k)} \tag{4.1}$$

and

$$(-1)^i e_k \int_{I_i^{(k)}} (f_k - Tf_k)\, dx \geqslant |||f_k - r_k|||. \tag{4.2}$$

Write $I_i^{(k)} = (a_i^{(k)}, b_i^{(k)})$ for $i = 1, ..., s$. By passing to subsequences, if necessary, assume that $a_i^{(k)} \to a_i$, $b_i^{(k)} \to b_i$ as $k \to \infty$ for each $i = 1, ..., s$, and $e_k = e$ for all k, where $e = 1$ or -1, fixed. Thus $a_1 < b_1 \leqslant a_2 < \cdots \leqslant a_s < b_s$.

It is shown that if $q > 0$ on $[a_i, b_i]$ for some i with $1 \leqslant i \leqslant s$, then there is a real $x_i \in (a_i, b_i)$ such that

$$(-1)^i e(\tilde{p} - \tilde{q} r_0)(x_i) \leqslant 0. \tag{4.3}$$

Suppose to the contrary that $(-1)^i e(\tilde{p} - \tilde{q} r_0) > 0$ on (a_i, b_i) and $q > 0$ on $[a_i, b_i]$. Then $Tf_k \to \tilde{p}/\tilde{q}$, uniformly, on $[a_i, b_i]$. Letting $k \to \infty$ in (4.1) and (4.2), one has that $(-1)^i e(f - r_0) = (-1)^i e(f - \tilde{p}/\tilde{q}) + (-1)^i e(\tilde{p}/\tilde{q} - r_0) > 0$ on (a_i, b_i) and

$$\begin{aligned} |||f - r_0||| &\geqslant (-1)^i e \int_{a_i}^{b_i} (f - r_0)\, dx \\ &\geqslant c + (-1)^i e \int_{a_i}^{b_i} (\tilde{p}/\tilde{q} - r_0)\, dx > c, \end{aligned}$$

which is a contradiction.

Now set $M := \{0, s\} \cup \{i : 1 \leqslant i \leqslant s,\ [a_i, b_i] \cap \{z_1, ..., z_v\} = \varnothing\} \equiv \{i_1 < \cdots < i_{\tilde{s}}\}$, $\tilde{M} := \{t : 1 \leqslant t \leqslant \tilde{s},\ i_{t+1} - i_t \text{ is odd}\}$, and $Z(a, b) = \sum_{a \leqslant z_j \leqslant b} s_j$ with $0 \leqslant a < b \leqslant 1$. Since z_j intersects at most two intervals in $\{[a_i, b_i] : i = 1, 2, ..., s\}$ and s_j is even provided $z_j \in (0, 1)$ for each $j = 1, 2, ..., v$, it follows that for $t = 1, ..., \tilde{s} - 1$,

$$Z(b_{i_t}, a_{i_{t+1}}) \geqslant \begin{cases} i_{t+1} - i_t & \text{if } i_{t+1} - i_t \text{ is even} \\ i_{t+1} - i_t - 1 & \text{if } i_{t+1} - i_t \text{ is odd.} \end{cases}$$

Therefore $s \leqslant Z(b_1, a_s) + \operatorname{card}(\tilde{M}) \leqslant Z(0, 1) + \operatorname{card}(\tilde{M})$. By the definition of $\tilde{M}$ and (4.3), $q_0 \tilde{p}/\tilde{q} - p_0$ has at least $s - Z(0, 1)$ weak sign changes in [0, 1] [5, Definition 13-1]. By Lemma of [1, p. 162] and Assertion of [4, p. 61] we have $p_0 = q_0 \tilde{p}/\tilde{q}$. Since $(\partial p_0 - n)(\partial q_0 - m) = 0$, it follows that $p/q = \tilde{p}/\tilde{q} = p_0/q_0$ and $q = \tilde{q} > 0$ on [0, 1]. Thus $Tf_k \to r_0$, uniformly, on [0, 1], which contradicts the assumption that $\|Tf_k - r_0\| \geqslant \varepsilon$.

(ii) $e < |||f - r_0|||$. By an analogous discussion on the alternating intervals of $f - r_0$, one can also obtain that $Tf_k \to r_0$, uniformly, as $k \to \infty$. The same contradiction as that in (i) is obtained.

Next we show that there exists a real $\delta_2 > 0$ such that every f with $\|f - f_0\| < \delta_2$ has a best approximation.

Assume $\|p_0\| + \|q_0\| = 1$. Let $2\varepsilon_1 = \inf_{0 \leqslant x \leqslant 1} q_0(x) > 0$. We claim that there exists an $\varepsilon_2 > 0$ such that

$$\left.\begin{array}{l} \|p\| + \|q\| = 1 \\ r = p/q \in R_m^n \\ \|r - r_0\| < \varepsilon_2 \end{array}\right\} \Rightarrow \|q - q_0\| < \varepsilon_1 .$$

Otherwise there exists a sequence $\{r_k = p_k/q_k\}$ in R_m^n with $\|p_k\| + \|q_k\| = 1$, $\|q_k - q_0\| \geqslant \varepsilon_1$ for each k, and $r_k \to r_0$ as $k \to \infty$. By passing to subsequences, if necessary, assume that $p_k \to p$ and $q_k \to q$ as $k \to \infty$. Then $p = qr_0$ and by Lemma 2 of [1, p. 165], $p = p_0$ and $q = q_0$, a contradiction.

Now a real δ_2 can be chosen so that for every f with $\|f - f_0\| < \delta_2$, its best approximation r (if it exists) satisfies that $\|r - r_0\| < \varepsilon_2$. Write $r = p/q$ with $\|p\| + \|q\| = 1$. Thus $\|q - q_0\| < \varepsilon_1$ and $q(x) > \varepsilon_1$ on $[0, 1]$. Therefore our search for r can be confined to the set

$$G := \{p/q : p/q \in R_m^n, q > \varepsilon_1\}.$$

It is elementary to show that G is compact and f has a best approximation from G (and thus from R_m^n).

$\delta = \min\{\delta_1, \delta_2\}$ is just what is needed in the theorem. The proof is completed.

If we consider the "continuity" of the operator T with respect to the measure $|||\cdot|||$ in the sense: given $f_0 \in C[0, 1]$, T is continuous at f_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|||Tf - Tf_0||| < \varepsilon$ whenever $|||f - f_0||| < \delta$, we can obtain the following result.

THEOREM 4.2. *The operator T is "discontinuous" everywhere in $C[0, 1]$ with respect to $|||\cdot|||$.*

Proof. Assume that $f_0 \in C[0, 1]$ has a best approximation Tf_0 from R_m^n. Write $Tf_0 = p_0/q_0$. By Theorem 2.3 there exist $s = \max\{\partial p_0 + m, \partial q_0 + n\} + 2$ open intervals $I_1 < \cdots < I_s$ such that for $i = 1, 2, ..., s$,

$$(-1)^i e(f_0 - Tf_0) \geqslant 0 \qquad \text{on } I_i$$

$$(-1)^i e \int_{I_i} (f_0 - Tf_0)\, dx \geqslant |||f_0 - Tf_0|||.$$

For $c>0$ sufficiently small we can choose s closed intervals $\tilde{I}_1 < \cdots < \tilde{I}_s$ such that $(-1)^i e(f_0 - Tf_0 - c/q_0) \geqslant 0$ on $\tilde{I}_i$ and $f_0 - Tf_0 - c/q_0 = 0$ at both endpoints of $\tilde{I}_i$ for each $i=1, 2, ..., s$. Let $\tilde{c} = \inf\{ \|\| f_0 - Tf_0 - c/q_0 \|\|_{\tilde{I}_i} : 1 \leqslant i \leqslant s\}$ and $\tilde{I}_i = [a_i, b_i]$. Now for every $\delta>0$, define a function in $C[0, 1]$ such that for $i=1, 2, ..., s$,

$$(-1)^i e(f - Tf_0 - c/q_0) \geqslant 0 \qquad \text{on } \tilde{I}_i,$$

$$\|\, f - Tf_0 - c/q_0 \|\|_{\tilde{I}_i} = \tilde{c},$$

$$(f - Tf_0 - c/q_0)(a_i) = (f - Tf_0 - c/q_0)(b_i) = 0,$$

and $\|\| f - Tf_0 - c/q_0 \|\| = \tilde{c}$, $\|\| f - f_0 \|\| < \delta$. This function can be constructed directly (some oscillating function between f_0 and $Tf_0 + c/q_0$ will meet the above requirements). Thus $\tilde{I}_1, ..., \tilde{I}_s$ are s alternating intervals of $f - Tf_0 - c/q_0$. Since $\max\{\partial(p_0 + c) + m, \; \partial q_0 + n\} + 2 = s$, it follows that $Tf = (p_0 + c)/q_0$. However, $\|\, f - f_0 \|\| < \delta$ and $\|\| Tf - Tf_0 \|\| = c \, \|\| 1/q_0 \|\| > 0$. Hence the operator T is "discontinuous" at f_0. The proof is completed.

The "discontinuity" of best approximation from P_n with respect to $\|\| \cdot \|\|$ can also be obtained as a special case of Theorem 4.2 with $m=0$.

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